compact, its preimage in X must be compact. Each point y_i has precisely one preimage point x_i in X, and y possesses precisely one preimage point x, which must belong to W. Since $\{x, x_i\}$ is compact, by passage to a subsequence we may assume that x_i converges to a point $z \in X$. Then $f(x_i) \longrightarrow f(z)$; therefore since $f(x_i) \longrightarrow f(x)$, the injectivity of f implies that $z = x$. Now W is open; so since $x_i \longrightarrow x$, we conclude that, for large i, $x_i \in W$. This result contradicts the assumption $y_i \notin f(W)$. So $f(X)$ is indeed a manifold. It is now trivial to check that $f: X \longrightarrow f(X)$ is a diffeomorphism, for we now know f to be a local diffeomorphism from X to $f(X)$. Since it is bijective, the inverse f^{-1}: $f(X) \longrightarrow X$ is well defined as a set map. But locally f^{-1} is already known to be smooth. Q.E.D.

Of course, when X itself is a compact manifold, every map $f: X \longrightarrow Y$ is proper. Thus for compact manifolds, embeddings are just one-to-one immersions.

EXERCISES

1. Let A be a linear map of $\mathbf{R}^n$, and $b \in \mathbf{R}^n$. Show that the mapping $x \longrightarrow Ax + b$ is a diffeomorphism of $\mathbf{R}^n$ if and only if A is nonsingular.

*2. Suppose that Z is an l-dimensional submanifold of X and that $z \in Z$. Show that there exists a local coordinate system $\{x_1, \ldots, x_k\}$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{l+1} = 0, \ldots, x_k = 0$.

3. Let $f: \mathbf{R}^1 \longrightarrow \mathbf{R}^1$ be a local diffeomorphism. Prove that the image of f is an open interval and that, in fact, f maps $\mathbf{R}^1$ diffeomorphically onto this interval.

4. To contrast with Exercise 3, construct a local diffeomorphism $f: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ that is not a diffeomorphism onto its image. [HINT: Start with our example for $\mathbf{R}^1 \longrightarrow S^1$.]

5. Prove that a local diffeomorphism $f: X \longrightarrow Y$ is actually a diffeomorphism of X onto an open subset of Y, provided that f is one-to-one.

6. (a) If f and g are immersions, show that $f \times g$ is.
   (b) If f and g are immersions, show that $g \circ f$ is.
   (c) If f is an immersion, show that its restriction to any submanifold of its domain is an immersion.
   (d) When dim $X =$ dim Y, show that immersions $f: X \longrightarrow Y$ are the same as local diffeomorphisms.

7. (a) Check that $g: \mathbf{R}^1 \longrightarrow S^1$, $g(t) = (\cos 2\pi t, \sin 2\pi t)$, is, in fact, a local diffeomorphism.

(b) From Exercise 6, it follows that $G: \mathbf{R}^2 \longrightarrow S^1 \times S^1$, $G = g \times g$, is a local diffeomorphism. Also, if L is a line in $\mathbf{R}^2$, the restriction $G: L \longrightarrow S^1 \times S^1$ is an immersion. Prove that if L has irrational slope, G is one-to-one on L.

8. Check that the map

$$\mathbf{R}^1 \longrightarrow \mathbf{R}^2, \qquad t \longrightarrow \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2}\right),$$

is an embedding. Prove that its image is one nappe of the hyperbola $x^2 - y^2 = 1$.

***9.** (a) Let $x_1, \ldots, x_N$ be the standard coordinate functions on $\mathbf{R}^N$, and let X be a k-dimensional submanifold of $\mathbf{R}^N$. Prove that every point $x \in X$ has a neighborhood on which the restrictions of some k-coordinate functions $x_{i_1}, \ldots, x_{i_k}$ form a local coordinate system. [HINT: Let $e_1, \ldots, e_N$ be the usual basis for $\mathbf{R}^N$. As a linear algebra lemma, prove that the projection of $T_x(X)$ onto the subspace spanned by $e_{i_1}, \ldots, e_{i_k}$ is bijective for some choice of $i_1, \ldots, i_k$. Show that this implies that $(x_{i_1}, \ldots, x_{i_k})$ defines a local diffeomorphism of X into $\mathbf{R}^k$ at the point x.]

(b) For simplicity, assume that $x_1, \ldots, x_k$ form a local coordinate system on a neighborhood V of x in X. Prove that there are smooth functions $g_{k+1}, \ldots, g_N$ on an open set U in $\mathbf{R}^k$ such that V may be taken to be the set

$$\{(a_1, \ldots, a_k, g_{k+1}(a), \ldots, g_N(a)) \in R^N : \ a = (a_1, \ldots, a_k) \in U\}.$$

That is, if we define $g: U \longrightarrow \mathbf{R}^{N-k}$ by $g = (g_{k+1}, \ldots, g_N)$, then V equals the graph of g. Thus every manifold is locally expressible as a graph.

***10.** *Generalization of the Inverse Function Theorem:* Let $f: X \longrightarrow Y$ be a smooth map that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x : T_x(X) \longrightarrow T_{f(x)}(Y)$$

is an isomorphism. Then f maps Z diffeomorphically onto $f(Z)$. (Why?) Prove that f, in fact, maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y. Note that when Z is a single point, this specializes to the Inverse Function Theorem. [HINT: Prove that, by Exercise 5, you need only show f to be one-to-one on some neighborhood of Z. Now if f isn't so, construct sequences $\{a_i\}$ and $\{b_i\}$ in X both converging to a point $z \in Z$, with $a_i \neq b_i$ but $f(a_i) = f(b_i)$. Show that this contradicts the nonsingularity of df_z.]